Comment on "A neoclassical calculation of rotation profiles and comparison with DIII-D measurements"

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Abstract

The calculation presented in "A neoclassical calculation of rotation profiles and comparison with DIII-D measurements" by Stacey, Johnson, and Mandrekas, [*Physics of Plasmas*, **13**, (2006)], contains several errors, including the neglect of the toroidal electric field, an unphysical expression for the electrostatic potential, and an unevaluated relation among its parameters. An alternative formulation is discussed.

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The calculation presented in "A neoclassical calculation of rotation profiles and comparison with DIII-D measurements" by Stacey, Johnson, and Mandrekas [1], contains several errors, including the neglect of the toroidal electric field, an unphysical expression for the electrostatic potential, and an unevaluated relation among its parameters. These inaccurate and neglected expressions have been part of the Georgia Tech Fusion Research Center (GT-FRC) model for over fifteen years [2, 3] and hamper its ability to interpret the results of experimental investigation.

Reference [1] writes the toroidal equation of motion for species j as

$$n_j m_j \left[\left(\mathbf{V}_j \cdot \nabla \right) \mathbf{V}_j \right]_\phi + \left[\nabla \cdot \overleftrightarrow{\mathbf{\Pi}}_j \right]_\phi - M_{\phi j} - F_{\phi j} - n_j e_j \left(V_{rj} B_\theta + E_\phi \right) = 0 \, . \tag{1}$$

While the toroidal electric field is retained symbolically in Equation (27) thereof, in practice the term is unevaluated, as no Ohm's law (generalized or otherwise) appears in the presentation of the theory. Thus, its determination of the gyroviscous angular momentum transport frequency via the solution of Equation (31) thereof is cast in doubt, as the resulting force balance equation neglects a term omnipresent for inductive current drive. A consistent treatment of the toroidal electric field in a plasma of finite conductivity necessarily involves retention of the full electron equation of motion with finite electron mass. Following Sturrock [4], the conductivity tensor in the concentric circular flux surface approximation, $\mathbf{B}/B \equiv (0, b_\theta, b_\phi)$, may be written using coordinate rotation tensors as

$$\overleftrightarrow{\sigma}_{\{r,\theta,\phi\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & b_\phi & b_\theta \\ 0 & -b_\theta & b_\phi \end{bmatrix} \begin{bmatrix} \sigma_\perp & -\sigma_H & 0 \\ \sigma_H & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & b_\phi & -b_\theta \\ 0 & b_\theta & b_\phi \end{bmatrix} \, , \tag{2}$$

given component conductivities $\sigma_\parallel \sim n_e e^2 / m_e \nu_{ee}$, $\sigma_\perp = \sigma_\parallel / [1 + (\omega_e/\nu_{ee})^2]$, and $\sigma_H = \sigma_\perp (\omega_e/\nu_{ee})$, for (unsigned) electron gyrofrequency ω_e and collision frequency ν_{ee}. Thus, the toroidal electric field induces currents in the poloidal plane as well as the toroidal direction, $\mathbf{J}^{E_\phi} = \left(J_r^{E_\phi}, J_\theta^{E_\phi}, J_\phi^{E_\phi} \right) = \left(b_\theta \sigma_H, b_\phi b_\theta (\sigma_\parallel - \sigma_\perp), b_\phi^2 \sigma_\parallel + b_\theta^2 \sigma_\perp \right) E_\phi$. With experimental determination of $J_\phi = J_\phi^{E_\phi} + \sigma_\parallel V_r B_\theta$ from the equilibrium equation $\nabla p = \mathbf{J} \times \mathbf{B}$ and the single fluid velocity $\mathbf{V} = \sum_j n_j m_j \mathbf{V}_j / \sum_j n_j m_j$, one may determine $E_\phi = J_\phi^{E_\phi} / (b_\phi^2 \sigma_\parallel + b_\theta^2 \sigma_\perp)$. Note that the radial current (including terms neglected by Sturrock), $J_r \equiv \hat{r} \cdot \overleftrightarrow{\sigma} \cdot \mathbf{E} + \sigma_\parallel [\partial p_e / n_e e \partial r + (V_\theta B_\phi - V_\phi B_\theta)]$, must be zero at equilibrium.

Reference [1] writes the poloidal equation of motion for species j as

$$n_j m_j \left[\left(\mathbf{V}_j \cdot \nabla \right) \mathbf{V}_j \right]_\theta + \left[\nabla \cdot \overleftrightarrow{\mathbf{\Pi}}_j \right]_\theta + \frac{1}{r} \frac{\partial p_j}{\partial \theta} - F_{\theta j} + n_j e_j \left(V_{rj} B_\phi - E_\theta \right) = 0 \, , \tag{3}$$

and takes the poloidal electrostatic field on a flux surface at r as given by

$$E_\theta \equiv -\frac{1}{r}\frac{\partial\,\Phi(r)}{\partial\theta} \equiv -\frac{1}{r}\frac{\partial}{\partial\theta}\Phi^0(r)\,(1+\Phi^c\cos\theta+\Phi^s\sin\theta) = -\frac{\Phi^0(r)}{r}\,(\Phi^s\cos\theta-\Phi^c\sin\theta)\ , \tag{4}$$

where Φ is the electrostatic potential, indicating an expansion around $\Phi^0(r) \equiv -\int_a^r dr E_r^0 \neq 0$ for a last closed flux surface at $r = a$. The resulting evaluation of the flux surface unity, cosine, and sine moments of the electron poloidal equation of motion $Ŧ_e \partial n_e / r\partial\theta = -en_e E_\theta$ (other terms are assumed negligible at equilibrium), defined by the expressions $\langle A\rangle_{U,C,S} \equiv \oint d\theta\{1,\cos\theta,\sin\theta\}(1+\varepsilon\cos\theta)A/2\pi$, yields three equations which have only trivial solution. (The evaluation of these moments is best accomplished without recourse to the logarithmic derivative.) Specifically, for $n_e = n_e^0(1+n_e^c\cos\theta+n_e^s\sin\theta)$, we have the equations

$$U: \qquad \varepsilon Ŧ_e n_e^s = e\Phi^0(\varepsilon\Phi^s + n_e^c\Phi^s - n_e^s\Phi^c)\ , \tag{5}$$

$$C: \qquad Ŧ_e n_e^s = e\Phi^0(4\Phi^s + 3\varepsilon n_e^c\Phi^s - \varepsilon n_e^s\Phi^c)/4\ , \tag{6}$$

$$S: \qquad Ŧ_e n_e^c = e\Phi^0(4\Phi^c + \varepsilon n_e^c\Phi^c - \varepsilon n_e^s\Phi^s)/4\ , \tag{7}$$

valid $\forall\ \varepsilon, n_e^c, n_e^s$. Independent variation thereof for finite (fixed) Φ^0 yields $\Phi^{c,s} \equiv 0$ for this overdetermined system, thus the poloidal electrostatic field in the GT-FRC model, which fails to consider the unity moment equation and the $O(\varepsilon)$ terms, must vanish. Failing to include the $O(\varepsilon)$ terms indicates expressions applicable only on the magnetic axis $r = 0$, where $\varepsilon \equiv r/R_0 = 0$ for $R_0 \neq \infty$, yet the GT-FRC model is commonly used to address the physics near the edge of the confinement region [5]. These equations may be linearized and have the formal solution $(1/\Phi^0, \Phi^c, \Phi^s) \equiv (0,0,0)$, which we interpret to mean exactly what it says, that they are solved when $\Phi^0(r) \equiv \infty$, displaying its unphysical definition. As non-vanishing $\Phi^{c,s}$ are an integral part of the development of the GT-FRC model, appearing in both the poloidal and toroidal equations of motion, the validity of its conclusions is in jeopardy.

Note in passing that from $n_e = \sum_{ions} z_i n_i$ and $\partial p_e/\partial\theta = -\sum_{ions}\partial p_i/\partial\theta$, which for two ion species $z_i = 1$ and $z_z \neq 1$ give us the pair of equations

$$n_e^{c,s} = n_i^{c,s} + z_z n_z^{c,s} = -(p_i^0/p_e^0)n_i^{c,s} - (p_z^0/p_e^0)n_z^{c,s}\ , \tag{8}$$

the poloidal variations of the electron density, $n_e^{c,s}$, must vanish for a "pure" plasma consisting of a single (or single "effective") ion species but are supported in an "impure" plasma

consisting of multiple distinct ion species, when the electron and ion temperatures assume no poloidal variation as in the GT-FRC model. A single ion species plasma may support poloidal density asymmetries given by the poloidal dependence of the ion temperature, $\partial \log n_i/\partial\theta = -\partial \log(T_e + T_i)/\partial\theta$, which may also impact the multi-species treatment. The rightmost equality of Equations (8), providing an additional relation between $n_i^{c,s}$ and $n_z^{c,s}$, is neglected in the GT-FRC model, which treats the ion and impurity density asymmetries as independent free parameters.

An alternative treatment of the problem begins with the consideration of the poloidal line integral of the electric field, $\oint d\hat{l}\cdot\mathbf{E} \equiv \oint rd\theta E_\theta = 0$ at equilibrium. (Note that a non-vanishing radial electrostatic field without poloidal variation demands the existence of *no* poloidal electrostatic field, else the poloidal variation to the potential ruins the poloidal symmetry of the radial field.) Examining the expression of the leading contender for the poloidal electrostatic field [6] evaluated from the equation of motion,

$$E_\theta = \frac{\langle E_\phi B_\phi/B_\theta\rangle B^2}{\langle B^2/B_\theta\rangle B_\theta} - \frac{E_\phi B_\phi}{B_\theta} + RB_\phi p'\eta_\parallel \left(\frac{\langle 1/B_\theta\rangle B^2}{\langle B^2/B_\theta\rangle B_\theta} - \frac{1}{B_\theta}\right) , \tag{9}$$

one may put it in the form $E_\theta = E_\theta^{(2)}[2\varepsilon\cos\theta - (\varepsilon^2/2)\cos 2\theta]$, when the Shafranov shift is neglected, as in the concentric circular flux surface approximation of the GT-FRC model (and others), by requiring $\oint d\theta E_\theta = 0$. Inserting that expression into the electron poloidal equation of motion and taking the flux surface Fourier moments yields three equations which have a nontrivial solution only when expanded to order $O(\varepsilon^3)$, given by

$$\begin{bmatrix} n_e^c \\ n_e^s \\ E_\theta^{(2)} \end{bmatrix} = \begin{bmatrix} \varepsilon^3/(6\varepsilon^2 - 8) \\ \pm\varepsilon\sqrt{3\varepsilon^4 - 168\varepsilon^2 + 192}/(18\varepsilon^2 - 24) \\ \pm 4(T_e/eR_0)/\varepsilon\sqrt{3\varepsilon^4 - 168\varepsilon^2 + 192} \end{bmatrix} , \tag{10}$$

thus the presence of a poloidal electrostatic field of that form should be accompanied by a potentially measurable shift in the electron density profile. Note that the derivation immediately preceding is slightly inconsistent, as the associated electrostatic potential $\Phi \propto 2\varepsilon\sin\theta - (\varepsilon^2/4)\sin 2\theta$ is of the correct harmonic form $-\nabla^2_{axi}\Phi \equiv -(\partial^2/\partial Z^2 + \partial^2/\partial R^2)\Phi \equiv -[\partial(r\partial/\partial r)/r\partial r + \partial^2/r^2\partial\theta^2]\Phi = 0$ for axial geometry [7, 8, 9, 10] as implied by the neglect of the Shafranov shift, yet the flux surface average is done in toroidal geometry. Resolution by taking the potential over to tokamak geometry is forthcoming [11].

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